Real case

The slides still use an older, somewhat more complex form of the problem. A simpler way is to directly deal with the residual equation

$$\mathrm{r} = \nabla \times \boldsymbol{H} = 0$$

(in iron).

In the finite element case, we are solving the discrete residual equation

$$\boldsymbol{r_i} = \int \mathrm{N_i} \cdot \nabla \times \boldsymbol{H} dV = 0$$

For all *i* in 1….number_of_nodes, where N_i are the shape functions. We remember that **B** is of course defined with the vector potential:

$$\boldsymbol{B} = a_i \nabla \times N_i + a_1 \nabla \times N_2 + \cdots$$

In the finite element case, the entry (i, j) of the real Jacobian is then

$$\frac{\partial}{\partial a_j} \boldsymbol{r}_i = \int \frac{\partial}{\partial a_j} N_i \cdot (\nabla \times \boldsymbol{H}) dV$$

which is simplified into

$$\int N_i \cdot \nabla \times \left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}} \frac{\partial \boldsymbol{B}}{\partial a_j} \right) dV$$

with the chain rule of differentiation:

$$\frac{\partial \boldsymbol{H}}{\partial a_j} = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}} \frac{\partial \boldsymbol{B}}{\partial a_j}$$

The expression is then further simplified by noting that

$$\frac{\partial \boldsymbol{B}}{\partial a_j} = \nabla \times N_j.$$

Finally, the expression

$$\int N_i \cdot \nabla \times \left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}} \nabla \times N_j \right) dV$$

is simplified (see below) into the more-familiar curl-curl form

$$\int (\nabla \times N_i) \cdot \left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}} \nabla \times N_j\right) dV.$$

(The curl-curl manipulation is done with the identity (see Potential Formulations in Magnetics, http://maxwell.sze.hu/docs/C4.pdf page 80 or so)

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v},$$

Using

$$v = N_i$$

and

$$u = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}} \nabla \times N_j$$

) See the Appendix for a more thorough derivation.

Handling the differential reluctivity term

Now, the only difficulty left is evaluating the vector-by-vector derivative (for more info, the Wikipedia page can help)

$$\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}} = \begin{bmatrix} \frac{\partial H_x}{\partial B_x} & \frac{\partial H_x}{\partial B_y} \\ \frac{\partial H_y}{\partial B_x} & \frac{\partial H_y}{\partial B_y} \end{bmatrix}$$

For isotropic materials with no hysteresis, a helpful approach is to use the reluctivity written as a function of the square of the flux density, yielding e.g.

$$\frac{\partial H_x}{\partial B_x} = \frac{\partial}{\partial B_x}\left(\nu(B^2) B_x\right) = \nu \frac{\partial B_x}{\partial B_x} + \frac{\partial \nu(B^2)}{\partial B} B_x = \nu + \left(\frac{\partial \nu}{\partial B^2}\frac{\partial B^2}{\partial B_x}\right) B_x \,.$$

where the second form is obtained using the derivative-of-product formula. The final form is then obtained by treating the derivative-of-reluctivity term with the chain rule of differentiation. The reluctivity derivative $\frac{\partial \nu}{\partial B^2}$ is normally known directly as such from an interpolation table.

The flux density derivative is simplified into

$$\frac{\partial B^2}{\partial B_x} = \frac{\partial\left(B_x^2 + B_y^2\right)}{\partial B_x} = 2B_x$$

In the end, we thus have

$$\frac{\partial H_x}{\partial B_x} = \nu + B_x \frac{\partial \nu}{\partial B^2} 2B_x$$

For the dx/dy cross-term, the first term on the rhs disappears, yielding

$$\frac{\partial H_x}{\partial B_y} = 2B_x B_y \frac{\partial \nu}{\partial B^2}$$

Complex case

The complex case is analysed somewhat similarly, by splitting the residual into real and imaginary parts. The main differences are seen in the differential reluctivity tensor, as **H** now depends on both the real and imaginary components of **B**

$$\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{B}^R, \boldsymbol{B}^I)$$

Then, we see e.g. how the real part of the residual depends on both the real and complex part of the vector potential:

$$\frac{\partial}{\partial a_j^R} \boldsymbol{r}_i^R = \int N_i \cdot (\nabla \times \frac{\partial}{\partial a_j^R} \boldsymbol{H}(\boldsymbol{B}^{\boldsymbol{R}}, \boldsymbol{B}^I) dV$$

Now, as the real part of B only depends on the real part of the vector potential, we get

$$\frac{\partial}{\partial a_j^R} \boldsymbol{r}_i^R = \int N_i \cdot \left(\nabla \times \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}^R} \frac{\partial}{\partial a_j^R} \boldsymbol{B}^R\right) dV = \int N_i \cdot (\nabla \times \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}^R} \nabla \times N_j) dV$$

Similarly, we get for the off-diagonal block of the Jacobian for example

$$\frac{\partial}{\partial a_j^I} \boldsymbol{r}_i^R = \int N_i \cdot \left(\nabla \times \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{B}^I} \nabla \times N_j\right) dV.$$

For non-hysteretic isotropic material, we see another difference in the squared amplitude of B:

$$B^2 = B_x^{r,2} + B_y^{r,2} + B_x^{r,2} + B_y^{i,2}$$

Other than that, the treatment of the reluctivity tensor is similar to the real case.

APPENDIX: Derivation of the curl-curl equation

We begin with the equation

$$\boldsymbol{v} \cdot \nabla \times (\nu \nabla \times \boldsymbol{A})$$

Denote

$$\boldsymbol{u} = \nu \nabla \times \boldsymbol{A}$$

To reduce the expression into

$$\boldsymbol{v} \cdot \nabla \times \boldsymbol{u}$$

Using the identity, we get

$$\boldsymbol{v} \cdot \nabla \times \boldsymbol{u} = \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) + \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}$$

Taking the integral of the right-hand side term and using the Gauss theorem on the first term, we get

$$\int_V \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) + \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} dV = \int_{\partial V} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n} dS + \int_V \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} dV$$

After substituting back the definition of u, we get

$$\int_{\partial V} (\nu(\nabla \times \boldsymbol{A}) \times \boldsymbol{v}) \cdot \boldsymbol{n} dS + \int_V (\nu \nabla \times \boldsymbol{A}) \cdot (\nabla \times \boldsymbol{v}) dV$$

The latter term is the familiar curl-curl integral. Next, let's show that the boundary term disappears in typical problems.

For the boundary integral, we can again apply the vector triple product identity to get

$$\int_{\partial V} (\nu(\nabla \times \boldsymbol{A}) \times \boldsymbol{v}) \cdot \boldsymbol{n} dS = -\int_{\partial V} (\boldsymbol{n} \times \boldsymbol{v}) \cdot (\nu \nabla \times \boldsymbol{A}) dS$$

The latter term is by definition identically zero on the entire boundary (under typical conditions):

- On Dirichlet boundaries, we are only using such test functions **v** that are zero on the boundary. Thus, the first term of the integrand is equal to zero.
- On Neumann boundaries, *B* is by definition perpendicular to the boundary, thus parallel to **n**. On these boundaries, we can write $\nu \nabla \times A = c\boldsymbol{n}$ where c is some constant. Thus, the integrand is reduced to $(\boldsymbol{n} \times \boldsymbol{v}) \cdot c\boldsymbol{n}$ which is identically zero.